

# AB-CONTEXTS AND STABILITY FOR GORENSTEIN FLAT MODULES WITH RESPECT TO SEMIDUALIZING MODULES

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**ABSTRACT.** We investigate the properties of categories of  $G_C$ -flat  $R$ -modules where  $C$  is a semidualizing module over a commutative noetherian ring  $R$ . We prove that the category of all  $G_C$ -flat  $R$ -modules is part of a weak AB-context, in the terminology of Hashimoto. In particular, this allows us to deduce the existence of certain Auslander-Buchweitz approximations for  $R$ -modules of finite  $G_C$ -flat dimension. We also prove that two procedures for building  $R$ -modules from complete resolutions by certain subcategories of  $G_C$ -flat  $R$ -modules yield only the modules in the original subcategories.

## INTRODUCTION

Auslander and Bridger [1, 2] introduce the modules of finite G-dimension over a commutative noetherian ring  $R$ , in part, to identify a class of finitely generated  $R$ -modules with particularly nice duality properties with respect to  $R$ . They are exactly the  $R$ -modules which admit a finite resolution by modules of G-dimension 0. As a special case, the duality theory for these modules recovers the well-known duality theory for finitely generated modules over a Gorenstein ring.

This notion has been extended in several directions. For instance, Enochs, Jenda and Torrecillas [8, 10] introduce the Gorenstein projective modules and the Gorenstein flat modules; these are analogues of modules of G-dimension 0 for the non-finitely generated arena. Foxby [11], Golod [13] and Vasconcelos [25] focus on finitely generated modules, but consider duality with respect to a semidualizing module  $C$ . Recently, Holm and Jørgensen [17] have unified these approaches with the  $G_C$ -projective modules and the  $G_C$ -flat modules. For background and definitions, see Sections 1 and 2.

The purpose of this paper is to use cotorsion flat modules in order to further study the  $G_C$ -flat modules, which are more technically challenging to investigate than the  $G_C$ -projective modules. Cotorsion flat modules have been successfully used to investigate flat modules, for instance in the work of Xu [27], and this paper shows how they are similarly well-suited for studying the  $G_C$ -flat modules.

More specifically, an  $R$ -module is *C-flat C-cotorsion* when is isomorphic to an  $R$ -module of the form  $F \otimes_R C$  where  $F$  is flat and cotorsion. We let  $\widehat{\mathcal{F}_C^{\text{cot}}}(R)$  denote the category of all  $C$ -flat  $C$ -cotorsion  $R$ -modules, and we let  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}}(R)$  denote the category of all  $R$ -modules admitting a finite resolution by  $C$ -flat  $C$ -cotorsion  $R$ -modules. The first step of our analysis is carried out in Section 3 where we

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investigate the fundamental properties of these categories; see Theorem I(b) for some of the conclusions from this section.

Section 4 contains our analysis of the category of  $G_C$ -flat modules, denoted  $\mathcal{GF}_C(R)$ . This section culminates in the following theorem. In the terminology of Hashimoto [15], it says that the triple  $(\mathcal{GF}_C(R), \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$  satisfies the axioms for a weak AB-context. The proof of this result is in (4.9).

**Theorem I.** *Let  $C$  be a semidualizing  $R$ -module.*

- (a)  $\mathcal{GF}_C(R)$  is closed under extensions, kernels of epimorphisms and summands.
- (b)  $\widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$  is closed under cokernels of monomorphisms, extensions and summands, and  $\widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)} \subseteq \widehat{\mathcal{GF}_C(R)}$ .
- (c)  $\mathcal{F}_C^{\text{cot}}(R) = \mathcal{GF}_C(R) \cap \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ , and  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for  $\mathcal{GF}_C(R)$ .

In conjunction with [15, (1.12.10)], this result implies many of the conclusions of [3] for the triple  $(\mathcal{GF}_C(R), \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$ . For instance, we conclude that every module  $M$  of finite  $G_C$ -flat dimension fits in an exact sequence

$$0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$$

such that  $X$  is in  $\mathcal{GF}_C(R)$  and  $Y$  is in  $\widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ . Such “approximations” have been very useful, for instance, in the study of modules of finite  $G$ -dimension. See Corollary 4.10 for this and other conclusions.

In Section 5 we apply these techniques to continue our study of stability properties of Gorenstein categories, initiated in [23]. For each subcategory  $\mathcal{X}$  of the category of  $R$ -modules, let  $\mathcal{G}^1(\mathcal{X})$  denote the category of all  $R$ -modules isomorphic to  $\text{Coker}(\partial_1^X)$  for some exact complex  $X$  in  $\mathcal{X}$  such that the complexes  $\text{Hom}_R(X', X)$  and  $\text{Hom}_R(X, X')$  are exact for each module  $X'$  in  $\mathcal{X}$ . This definition is a modification of the construction of  $G_C$ -projective  $R$ -modules. Inductively, set  $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$  for each  $n \geq 1$ . The techniques of this paper allow us to prove the following  $G_C$ -flat versions of some results of [23]; see Corollary 5.10 and Theorem 5.14.

**Theorem II.** *Let  $C$  be a semidualizing  $R$ -module and let  $n \geq 1$ .*

- (a) *We have  $\mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ .*
- (b) *If  $\dim(R) < \infty$ , then  $\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$ .*

Here  $\mathcal{B}_C(R)$  is the Bass class associated to  $C$ , and  $\mathcal{F}_C(R)^\perp$  is the category of all  $R$ -modules  $N$  such that  $\text{Ext}_R^{\geq 1}(F \otimes_R C, N) = 0$  for each flat  $R$ -module  $F$ . In particular, when  $C = R$  this result yields  $\mathcal{G}^n(\mathcal{GF}(R)) = \mathcal{GF}(R)$  and, when  $\dim(R)$  is finite,  $\mathcal{G}^n(\mathcal{F}^{\text{cot}}(R)) = \mathcal{GF}(R) \cap \mathcal{F}(R)^\perp$ .

## 1. MODULES, COMPLEXES AND RESOLUTIONS

We begin with some notation and terminology for use throughout this paper.

**Definition 1.1.** Throughout this work  $R$  is a commutative noetherian ring and  $\mathcal{M}(R)$  is the category of  $R$ -modules. We use the term “subcategory” to mean a “full, additive subcategory  $\mathcal{X} \subseteq \mathcal{M}(R)$  such that, for all  $R$ -modules  $M$  and  $N$ , if  $M \cong N$  and  $M \in \mathcal{X}$ , then  $N \in \mathcal{X}$ .” Write  $\mathcal{P}(R)$ ,  $\mathcal{F}(R)$  and  $\mathcal{I}(R)$  for the subcategories of projective, flat and injective  $R$ -modules, respectively.

**Definition 1.2.** We fix subcategories  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{W}$ , and  $\mathcal{V}$  of  $\mathcal{M}(R)$  such that  $\mathcal{W} \subseteq \mathcal{X}$  and  $\mathcal{V} \subseteq \mathcal{Y}$ . Write  $\mathcal{X} \perp \mathcal{Y}$  if  $\text{Ext}_R^{\geq 1}(X, Y) = 0$  for each  $X \in \mathcal{X}$  and each  $Y \in \mathcal{Y}$ . For an  $R$ -module  $M$ , write  $M \perp \mathcal{Y}$  (resp.,  $\mathcal{X} \perp M$ ) if  $\text{Ext}_R^{\geq 1}(M, Y) = 0$  for each  $Y \in \mathcal{Y}$  (resp., if  $\text{Ext}_R^{\geq 1}(X, M) = 0$  for each  $X \in \mathcal{X}$ ). Set

$$\mathcal{X}^\perp = \text{the subcategory of } R\text{-modules } M \text{ such that } \mathcal{X} \perp M.$$

We say  $\mathcal{W}$  is a *cogenerator* for  $\mathcal{X}$  if, for each  $X \in \mathcal{X}$ , there is an exact sequence

$$0 \rightarrow X \rightarrow W \rightarrow X' \rightarrow 0$$

such that  $W \in \mathcal{W}$  and  $X' \in \mathcal{X}$ ; and  $\mathcal{W}$  is an *injective cogenerator* for  $\mathcal{X}$  if  $\mathcal{W}$  is a cogenerator for  $\mathcal{X}$  and  $\mathcal{X} \perp \mathcal{W}$ . The terms *generator* and *projective generator* are defined dually.

We say that  $\mathcal{X}$  is *closed under extensions* when, for every exact sequence

$$(*) \quad 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

if  $M', M'' \in \mathcal{X}$ , then  $M \in \mathcal{X}$ . We say that  $\mathcal{X}$  is *closed under kernels of monomorphisms* when, for every exact sequence  $(*)$ , if  $M', M \in \mathcal{X}$ , then  $M'' \in \mathcal{X}$ . We say that  $\mathcal{X}$  is *closed under cokernels of epimorphisms* when, for every exact sequence  $(*)$ , if  $M, M'' \in \mathcal{X}$ , then  $M' \in \mathcal{X}$ . We say that  $\mathcal{X}$  is *closed under summands* when, for every exact sequence  $(*)$ , if  $M \in \mathcal{X}$  and  $(*)$  splits, then  $M', M'' \in \mathcal{X}$ . We say that  $\mathcal{X}$  is *closed under products* when, for every set  $\{M_\lambda\}_{\lambda \in \Lambda}$  of modules in  $\mathcal{X}$ , we have  $\prod_{\lambda \in \Lambda} M_\lambda \in \mathcal{X}$ .

**Definition 1.3.** We employ the notation from [5] for  $R$ -complexes. In particular,  $R$ -complexes are indexed homologically

$$M = \cdots \xrightarrow{\partial_{n+1}^M} M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} \cdots$$

with  $n$ th homology module denoted  $H_n(M)$ . We frequently identify  $R$ -modules with  $R$ -complexes concentrated in degree 0.

Let  $M, N$  be  $R$ -complexes. For each integer  $i$ , let  $\Sigma^i M$  denote the complex with  $(\Sigma^i M)_n = M_{n-i}$  and  $\partial_n^{\Sigma^i M} = (-1)^i \partial_{n-i}^M$ . Let  $\text{Hom}_R(M, N)$  and  $M \otimes_R N$  denote the associated Hom complex and tensor product complex, respectively. A morphism  $\alpha: M \rightarrow N$  is a *quasiisomorphism* when each induced map  $H_n(\alpha): H_n(M) \rightarrow H_n(N)$  is bijective. Quasiisomorphisms are designated by the symbol  $\simeq$ .

The complex  $M$  is  $\text{Hom}_R(\mathcal{X}, -)$ -*exact* if the complex  $\text{Hom}_R(X, M)$  is exact for each  $X \in \mathcal{X}$ . Dually, the complex  $M$  is  $\text{Hom}_R(-, \mathcal{X})$ -*exact* if  $\text{Hom}_R(M, X)$  is exact for each  $X \in \mathcal{X}$ , and  $M$  is  $- \otimes_R \mathcal{X}$ -*exact* if  $M \otimes_R X$  is exact for each  $X \in \mathcal{X}$ .

**Definition 1.4.** When  $X_{-n} = 0 = H_n(X)$  for all  $n > 0$ , the natural morphism  $X \rightarrow H_0(X) = M$  is a quasiisomorphism, that is, the following sequence is exact

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \rightarrow M \rightarrow 0.$$

In this event,  $X$  is an  $\mathcal{X}$ -*resolution* of  $M$  if each  $X_n$  is in  $\mathcal{X}$ , and  $X^+$  is the *augmented  $\mathcal{X}$ -resolution* of  $M$  associated to  $X$ . We write “projective resolution” in lieu of “ $\mathcal{P}$ -resolution”, and we write “flat resolution” in lieu of “ $\mathcal{F}$ -resolution”. The  $\mathcal{X}$ -*projective dimension* of  $M$  is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

The modules of  $\mathcal{X}$ -projective dimension 0 are the nonzero modules of  $\mathcal{X}$ . We set

$$\text{res } \widehat{\mathcal{X}} = \text{the subcategory of } R\text{-modules } M \text{ with } \mathcal{X}\text{-pd}_R(M) < \infty.$$

One checks easily that  $\text{res } \widehat{\mathcal{X}}$  is additive and contains  $\mathcal{X}$ . Following established conventions, we set  $\text{pd}_R(M) = \mathcal{P}\text{-pd}_R(M)$  and  $\text{fd}_R(M) = \mathcal{F}\text{-pd}_R(M)$ .

The term  $\mathcal{Y}$ -coresolution is defined dually. The  $\mathcal{Y}$ -injective dimension of  $M$  is denoted  $\mathcal{Y}\text{-id}_R(M)$ , and the *augmented  $\mathcal{Y}$ -coresolution* associated to a  $\mathcal{Y}$ -coresolution  $Y$  is denoted  ${}^+Y$ . We write “injective resolution” for “ $\mathcal{I}$ -coresolution”, and we set

$$\text{cores } \widehat{\mathcal{Y}} = \text{the subcategory of } R\text{-modules } N \text{ with } \mathcal{Y}\text{-id}_R(N) < \infty$$

which is additive and contains  $\mathcal{Y}$ .

**Definition 1.5.** A  $\mathcal{Y}$ -coresolution  $Y$  is  $\mathcal{X}$ -proper if the augmented resolution  ${}^+Y$  is  $\text{Hom}_R(-, \mathcal{X})$ -exact. We set

$$\text{cores } \widetilde{\mathcal{Y}} = \text{the subcategory of } R\text{-modules admitting a } \mathcal{Y}\text{-proper } \mathcal{Y}\text{-coresolution.}$$

One checks readily that  $\text{cores } \widetilde{\mathcal{Y}}$  is additive and contains  $\mathcal{Y}$ . The term  $\mathcal{Y}$ -proper  $\mathcal{X}$ -resolution is defined dually.

**Definition 1.6.** An  $\mathcal{X}$ -precover of an  $R$ -module  $M$  is an  $R$ -module homomorphism  $\varphi: X \rightarrow M$  where  $X \in \mathcal{X}$  such that, for each  $X' \in \mathcal{X}$ , the homomorphism  $\text{Hom}_R(X', \varphi): \text{Hom}_R(X', X) \rightarrow \text{Hom}_R(X', M)$  is surjective. An  $\mathcal{X}$ -precover  $\varphi: X \rightarrow M$  is an  $\mathcal{X}$ -cover if, every endomorphism  $f: X \rightarrow X$  such that  $\varphi = \varphi f$  is an automorphism. The terms *preenvelope* and *envelope* are defined dually.

The next three lemmata have standard proofs; see [3, proofs of (2.1) and (2.3)].

**Lemma 1.7.** Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequence of  $R$ -modules.

- (a) If  $M_3 \perp \mathcal{W}$ , then  $M_1 \perp \mathcal{W}$  if and only if  $M_2 \perp \mathcal{W}$ . If  $M_1 \perp \mathcal{W}$  and  $M_2 \perp \mathcal{W}$ , then  $M_3 \perp \mathcal{W}$  if and only if the given sequence is  $\text{Hom}_R(-, \mathcal{W})$ -exact.
- (b) If  $\mathcal{V} \perp M_1$ , then  $\mathcal{V} \perp M_2$  if and only if  $\mathcal{V} \perp M_3$ . If  $\mathcal{V} \perp M_2$  and  $\mathcal{V} \perp M_3$ , then  $\mathcal{V} \perp M_1$  if and only if the given sequence is  $\text{Hom}_R(\mathcal{V}, -)$ -exact.
- (c) If  $\text{Tor}_{\geq 1}^R(M_3, \mathcal{V}) = 0$ , then  $\text{Tor}_{\geq 1}^R(M_1, \mathcal{V}) = 0$  if and only if  $\text{Tor}_{\geq 1}^R(M_2, \mathcal{V}) = 0$ . If  $\text{Tor}_{\geq 1}^R(M_1, \mathcal{V}) = 0 = \text{Tor}_{\geq 1}^R(M_2, \mathcal{V})$ , then  $\text{Tor}_{\geq 1}^R(M_3, \mathcal{V}) = 0$  if and only if the given sequence is  $- \otimes_R \mathcal{V}$ -exact.  $\square$

**Lemma 1.8.** If  $\mathcal{X} \perp \mathcal{Y}$ , then  $\mathcal{X} \perp \text{res } \widehat{\mathcal{Y}}$  and  $\text{cores } \widehat{\mathcal{X}} \perp \mathcal{Y}$ .  $\square$

**Lemma 1.9.** Let  $X$  be an exact  $R$ -complex.

- (a) Assume  $X_i \perp \mathcal{V}$  for all  $i$ . If  $X$  is  $\text{Hom}_R(-, \mathcal{V})$ -exact, then  $\text{Ker}(\partial_i^X) \perp \mathcal{V}$  for all  $i$ . Conversely, if  $\text{Ker}(\partial_i^X) \perp \mathcal{V}$  for all  $i$  or if  $X_i = 0$  for all  $i \ll 0$ , then  $X$  is  $\text{Hom}_R(-, \mathcal{V})$ -exact.
- (b) Assume  $\mathcal{V} \perp X_i$  for all  $i$ . If  $X$  is  $\text{Hom}_R(\mathcal{V}, -)$ -exact, then  $\mathcal{V} \perp \text{Ker}(\partial_i^X)$  for all  $i$ . Conversely, if  $\mathcal{V} \perp \text{Ker}(\partial_i^X)$  for all  $i$  or if  $X_i = 0$  for all  $i \gg 0$ , then  $X$  is  $\text{Hom}_R(\mathcal{V}, -)$ -exact.
- (c) Assume  $\text{Tor}_{\geq 1}^R(X_i, \mathcal{V}) = 0$  for all  $i$ . If the complex  $X$  is  $- \otimes_R \mathcal{V}$ -exact, then  $\text{Tor}_{\geq 1}^R(\text{Ker}(\partial_i^X), \mathcal{V}) = 0$  for all  $i$ . Conversely, if  $\text{Tor}_{\geq 1}^R(\text{Ker}(\partial_i^X), \mathcal{V}) = 0$  for all  $i$  or if  $X_i = 0$  for all  $i \ll 0$ , then  $X$  is  $- \otimes_R \mathcal{V}$ -exact.  $\square$

A careful reading of the proofs of [23, (2.1),(2.2)] yields the next result.

**Lemma 1.10.** Assume that  $\mathcal{W}$  is an injective cogenerator for  $\mathcal{X}$ . If  $M$  has an  $\mathcal{X}$ -coresolution that is  $\mathcal{W}$ -proper and  $M \perp \mathcal{W}$ , then  $M$  is in  $\text{cores } \widetilde{\mathcal{W}}$ .  $\square$

## 2. CATEGORIES OF INTEREST

This section contains definitions of and basic facts about the categories to be investigated in this paper.

**Definition 2.1.** An  $R$ -module  $M$  is *cotorsion* if  $\mathcal{F}(R) \perp M$ . We set

$\mathcal{F}^{\text{cot}}(R)$  = the subcategory of flat cotorsion  $R$ -modules.

**Definition 2.2.** The *Pontryagin dual* or *character module* of an  $R$ -module  $M$  is the  $R$ -module  $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ .

One implication in the following lemma is from [27, (3.1.4)], and the others are established similarly.

**Lemma 2.3.** *Let  $M$  be an  $R$ -module.*

- (a) *The Pontryagin dual  $M^*$  is  $R$ -flat if and only if  $M$  is  $R$ -injective.*
- (b) *The Pontryagin dual  $M^*$  is  $R$ -injective if and only if  $M$  is  $R$ -flat.* □

Semidualizing modules, defined next, form the basis for our categories of interest.

**Definition 2.4.** A finitely generated  $R$ -module  $C$  is *semidualizing* if the natural homothety morphism  $R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism and  $\text{Ext}_R^{\geq 1}(C, C) = 0$ . An  $R$ -module  $D$  is *dualizing* if it is semidualizing and has finite injective dimension.

Let  $C$  be a semidualizing  $R$ -module. We set

$\mathcal{P}_C(R)$  = the subcategory of modules  $P \otimes_R C$  where  $P$  is  $R$ -projective

$\mathcal{F}_C(R)$  = the subcategory of modules  $F \otimes_R C$  where  $F$  is  $R$ -flat

$\mathcal{F}_C^{\text{cot}}(R)$  = the subcategory of modules  $F \otimes_R C$  where  $F$  is flat and cotorsion

$\mathcal{I}_C(R)$  = the subcategory of modules  $\text{Hom}_R(C, I)$  where  $I$  is  $R$ -injective.

Modules in  $\mathcal{P}_C(R)$ ,  $\mathcal{F}_C(R)$ ,  $\mathcal{F}_C^{\text{cot}}(R)$  and  $\mathcal{I}_C(R)$  are called  *$C$ -projective*,  *$C$ -flat*,  *$C$ -flat  $C$ -cotorsion*, and  *$C$ -injective*, respectively. An  $R$ -module  $M$  is  *$C$ -cotorsion* if  $\mathcal{F}_C(R) \perp M$ .

**Remark 2.5.** We justify the terminology “ $C$ -flat  $C$ -cotorsion” in Lemma 3.3 where we show that  $M$  is  $C$ -flat  $C$ -cotorsion if and only if it is  $C$ -flat and  $C$ -cotorsion.

The following categories were introduced by Foxby [12], Avramov and Foxby [4], and Christensen [6], though the idea goes at least back to Vasconcelos [25].

**Definition 2.6.** Let  $C$  be a semidualizing  $R$ -module. The *Auslander class* of  $C$  is the subcategory  $\mathcal{A}_C(R)$  of  $R$ -modules  $M$  such that

- (1)  $\text{Tor}_{\geq 1}^R(C, M) = 0 = \text{Ext}_R^{\geq 1}(C, C \otimes_R M)$ , and
- (2) The natural map  $M \rightarrow \text{Hom}_R(C, C \otimes_R M)$  is an isomorphism.

The *Bass class* of  $C$  is the subcategory  $\mathcal{B}_C(R)$  of  $R$ -modules  $M$  such that

- (1)  $\text{Ext}_R^{\geq 1}(C, M) = 0 = \text{Tor}_{\geq 1}^R(C, \text{Hom}_R(C, M))$ , and
- (2) The natural evaluation map  $C \otimes_R \text{Hom}_R(C, M) \rightarrow M$  is an isomorphism.

**Fact 2.7.** Let  $C$  be a semidualizing  $R$ -module. The categories  $\mathcal{A}_C(R)$  and  $\mathcal{B}_C(R)$  are closed under extensions, kernels of epimorphisms and cokernels of monomorphism; see [18, Cor. 6.3]. The category  $\mathcal{A}_C(R)$  contains all modules of finite flat dimension and those of finite  $\mathcal{I}_C$ -injective dimension, and the category  $\mathcal{B}_C(R)$  contains all modules of finite injective dimension and those of finite  $\mathcal{F}_C$ -projective dimension by [18, Cors. 6.1 and 6.2].

Arguing as in [5, (3.2.9)], we see that  $M \in \mathcal{A}_C(R)$  if and only if  $M^* \in \mathcal{B}_C(R)$ , and  $M \in \mathcal{B}_C(R)$  if and only if  $M^* \in \mathcal{A}_C(R)$ . Similarly, we have  $M \in \mathcal{B}_C(R)$  if and only if  $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$  by [24, (2.8.a)]. From [18, Thm. 6.1] we know that every module in  $\mathcal{B}_C(R)$  has a  $\mathcal{P}_C$ -proper  $\mathcal{P}_C$ -resolution.

The next definitions are due to Holm and Jørgensen [17] in this generality.

**Definition 2.8.** Let  $C$  be a semidualizing  $R$ -module. A *complete  $\mathcal{I}_C\mathcal{I}$ -resolution* is a complex  $Y$  of  $R$ -modules satisfying the following:

- (1)  $Y$  is exact and  $\text{Hom}_R(\mathcal{I}_C, -)$ -exact, and
- (2)  $Y_i$  is  $C$ -injective when  $i \geq 0$  and  $Y_i$  is injective when  $i < 0$ .

An  $R$ -module  $H$  is  *$G_C$ -injective* if there exists a complete  $\mathcal{I}_C\mathcal{I}$ -resolution  $Y$  such that  $H \cong \text{Coker}(\partial_1^Y)$ , in which case  $Y$  is a *complete  $\mathcal{I}_C\mathcal{I}$ -resolution of  $H$* . We set

$$\mathcal{GI}_C(R) = \text{the subcategory of } G_C\text{-injective } R\text{-modules.}$$

In the special case  $C = R$ , we write  $\mathcal{GI}(R)$  in place of  $\mathcal{GI}_R(R)$ .

A *complete  $\mathcal{FF}_C$ -resolution* is a complex  $Z$  of  $R$ -modules satisfying the following.

- (1)  $Z$  is exact and  $-\otimes_R \mathcal{I}_C$ -exact.
- (2)  $Z_i$  is flat if  $i \geq 0$  and  $Z_i$  is  $C$ -flat if  $i < 0$ .

An  $R$ -module  $M$  is  *$G_C$ -flat* if there exists a complete  $\mathcal{FF}_C$ -resolution  $Z$  such that  $M \cong \text{Coker}(\partial_1^Z)$ , in which case  $Z$  is a *complete  $\mathcal{FF}_C$ -resolution of  $M$* . We set

$$\mathcal{GF}_C(R) = \text{the subcategory of } G_C\text{-flat } R\text{-modules.}$$

In the special case  $C = R$ , we set  $\mathcal{GF}(R) = \mathcal{GF}_R(R)$ , and  $\text{Gfd} = \mathcal{GF}$ -pd.

A *complete  $\mathcal{PP}_C$ -resolution* is a complex  $X$  of  $R$ -modules satisfying the following.

- (1)  $X$  is exact and  $\text{Hom}_R(-, \mathcal{P}_C)$ -exact.
- (2)  $X_i$  is projective if  $i \geq 0$  and  $X_i$  is  $C$ -projective if  $i < 0$ .

An  $R$ -module  $M$  is  *$G_C$ -projective* if there exists a complete  $\mathcal{PP}_C$ -resolution  $X$  such that  $M \cong \text{Coker}(\partial_1^X)$ , in which case  $X$  is a *complete  $\mathcal{PP}_C$ -resolution of  $M$* . We set

$$\mathcal{GP}_C(R) = \text{the subcategory of } G_C\text{-projective } R\text{-modules.}$$

**Fact 2.9.** Let  $C$  be a semidualizing  $R$ -module. Flat  $R$ -modules and  $C$ -flat  $R$ -modules are  $G_C$ -flat by [17, (2.8.c)]. It is straightforward to show that an  $R$ -module  $M$  is  $G_C$ -flat if and only if the following conditions hold:

- (1)  $M$  admits an augmented  $\mathcal{F}_C$ -coresolution that is  $-\otimes_R \mathcal{I}_C$ -exact, and
- (2)  $\text{Tor}_{\geq 1}^R(M, \mathcal{I}_C) = 0$ .

Let  $R \ltimes C$  denote the trivial extension of  $R$  by  $C$ , defined to be the  $R$ -module  $R \ltimes_R C = R \oplus C$  with ring structure given by  $(r, c)(r', c') = (rr', rc' + r'c)$ . Each  $R$ -module  $M$  is naturally an  $R \ltimes C$ -module via the natural surjection  $R \ltimes C \rightarrow R$ . Within this protocol we have  $M \in \mathcal{GI}_C(R)$  if and only if  $M \in \mathcal{GI}(R \ltimes C)$  and  $M \in \mathcal{GF}_C(R)$  if and only if  $M \in \mathcal{GF}(R \ltimes C)$  by [17, (2.13) and (2.15)]. Also [17, (2.16)] implies  $\mathcal{GF}_C\text{-pd}_R(M) = \text{Gfd}_{R \ltimes C}(M)$ .

The next definition, from [23], is modeled on the construction of  $\mathcal{GI}(R)$ .

**Definition 2.10.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{M}(R)$ . A *complete  $\mathcal{X}$ -resolution* is an exact complex  $X$  in  $\mathcal{X}$  that is  $\text{Hom}_R(\mathcal{X}, -)$ -exact and  $\text{Hom}_R(-, \mathcal{X})$ -exact.<sup>1</sup> Such a complex is a *complete  $\mathcal{X}$ -resolution of  $\text{Coker}(\partial_1^X)$* . We set

$$\mathcal{G}(\mathcal{X}) = \text{the subcategory of } R\text{-modules with a complete } \mathcal{X}\text{-resolution.}$$

<sup>1</sup>In the literature, these complexes are sometimes called “totally acyclic”.

Set  $\mathcal{G}^0(\mathcal{X}) = \mathcal{X}$ ,  $\mathcal{G}^1(\mathcal{X}) = \mathcal{G}(\mathcal{X})$  and  $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$  for  $n \geq 1$ .

**Fact 2.11.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{M}(R)$ . Using a resolution of the form  $0 \rightarrow X \rightarrow 0$ , one sees that  $\mathcal{X} \subseteq \mathcal{G}(\mathcal{X})$  and so  $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{G}^{n+1}(\mathcal{X})$  for each  $n \geq 0$ . If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{G}^n(\mathcal{I}_C(R)) = \mathcal{G}\mathcal{I}_C(R) \cap \mathcal{A}_C(R)$  for each  $n \geq 1$ ; see [23, (5.5)].

The final definition of this section is for use in the proof of Theorem II.

**Definition 2.12.** Let  $C$  be a semidualizing  $R$ -module, and let  $\mathcal{X}$  be a subcategory of  $\mathcal{M}(R)$ . A  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{X}$ -resolution is an exact complex  $X$  in  $\mathcal{X}$  that is  $\text{Hom}_R(\mathcal{P}_C, -)$ -exact and  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact. Such a complex is a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{X}$ -resolution of  $\text{Coker}(\partial_1^X)$ . We set

$\mathcal{H}_C(\mathcal{X})$  = the subcategory of  $R$ -modules with a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{X}$ -resolution.

Set  $\mathcal{H}_C^0(\mathcal{X}) = \mathcal{X}$ ,  $\mathcal{H}_C^1(\mathcal{X}) = \mathcal{H}_C(\mathcal{X})$  and  $\mathcal{H}_C^{n+1}(\mathcal{X}) = \mathcal{H}_C(\mathcal{H}_C^n(\mathcal{X}))$  for each  $n \geq 1$ .

**Remark 2.13.** Let  $C$  be a semidualizing  $R$ -module, and let  $\mathcal{X}$  be a subcategory of  $\mathcal{M}(R)$ . Let  $X$  be an exact complex in  $\mathcal{X}$  that is  $\text{Hom}_R(C, -)$ -exact and  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact. Hom-tensor adjointness implies that  $X$  is  $\text{Hom}_R(\mathcal{P}_C, -)$ -exact and hence a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{X}$ -resolution, as is the complex  $\Sigma^i X$  for each  $i \in \mathbb{Z}$ . It follows that  $\text{Coker}(\partial_i^X) \in \mathcal{H}_C(\mathcal{X})$  for each  $i$ .

Using a resolution of the form  $0 \rightarrow X \rightarrow 0$ , one sees that  $\mathcal{X} \subseteq \mathcal{H}_C(\mathcal{X})$  and so  $\mathcal{H}_C^n(\mathcal{X}) \subseteq \mathcal{H}_C^{n+1}(\mathcal{X})$  for each  $n \geq 0$ . Furthermore, if  $\mathcal{F}_C(R) \subseteq \mathcal{X}$ , then  $\mathcal{G}(\mathcal{X}) \subseteq \mathcal{H}_C(\mathcal{X})$  and so  $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{H}_C^n(\mathcal{X})$  for each  $n \geq 1$ .

### 3. MODULES OF FINITE $\mathcal{F}_C^{\text{cot}}$ -PROJECTIVE DIMENSION

This section contains the fundamental properties of the modules of finite  $\mathcal{F}_C^{\text{cot}}$ -projective dimension. The first two results allow us to deduce information for these modules from the modules of finite  $\mathcal{I}_C(R)$ -injective dimension.

**Lemma 3.1.** Let  $M$  be an  $R$ -module, and let  $C$  be a semidualizing  $R$ -module.

- (a) The Pontryagin dual  $M^*$  is  $C$ -flat if and only if  $M$  is  $C$ -injective.
- (b) The Pontryagin dual  $M^*$  is  $C$ -injective if and only if  $M$  is  $C$ -flat.
- (c) If  $\text{Tor}_{\geq 1}^R(C, M) = 0$ , then  $M^*$  is  $C$ -cotorsion.
- (d) If  $M$  is  $C$ -injective, then  $M^*$  is  $C$ -flat and  $C$ -cotorsion.

*Proof.* (a) Assume that  $M$  is  $C$ -injective, so there exists an injective  $R$ -module  $I$  such that  $M \cong \text{Hom}_R(C, I)$ . This yields the first isomorphism in the following sequence while the second is from Hom-evaluation [7, (0.3.b)]:

$$M^* \cong \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(C, I), \mathbb{Q}/\mathbb{Z}) \cong C \otimes_R \text{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}).$$

Since  $I$  is injective, Lemma 2.3(b) implies that  $\text{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$  is flat. Hence, the displayed isomorphisms imply that  $M^*$  is  $C$ -flat.

Conversely, assume that  $M^*$  is  $C$ -flat, so there exists a flat  $R$ -module  $F$  such that  $M^* \cong F \otimes_R C$ . As  $F$  is flat it is in  $\mathcal{A}_C(R)$ , and this yields the first isomorphism in the next sequence, while the third isomorphism is Hom-tensor adjointness

$$F \cong \text{Hom}_R(C, F \otimes_R C) \cong \text{Hom}_R(C, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(C \otimes_R M, \mathbb{Q}/\mathbb{Z}).$$

This module is flat, and so Lemma 2.3(a) implies that  $C \otimes_R M$  is injective. From [18, Thm. 1] we conclude that  $M$  is  $C$ -injective.

(b) This is proved similarly.

(c) Let  $P$  be a projective resolution of  $M$ . Our Tor-vanishing hypothesis implies that there is a quasiisomorphism  $C \otimes_R P \simeq C \otimes_R M$ . For each flat  $R$ -module  $F$ , this yields a quasiisomorphism

$$F \otimes_R C \otimes_R P \simeq F \otimes_R C \otimes_R M.$$

Because  $\mathbb{Q}/\mathbb{Z}$  is injective over  $\mathbb{Z}$ , this provides the third quasiisomorphism in the next sequence, while the second quasiisomorphism is Hom-tensor adjointness

$$\begin{aligned} (*) \quad \text{Hom}_R(F \otimes_R C, P^*) &\simeq \text{Hom}_R(F \otimes_R C, \text{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})) \\ &\simeq \text{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R P, \mathbb{Q}/\mathbb{Z}) \\ &\simeq \text{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Since  $\mathbb{Q}/\mathbb{Z}$  is injective over  $\mathbb{Z}$ , there are quasiisomorphisms

$$M^* \simeq \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}) \simeq P^*.$$

By Lemma 2.3(a), it follows that  $P^*$  is an injective resolution of  $M^*$  over  $R$ . In particular, taking cohomology in the displayed sequence  $(*)$  yields isomorphisms

$$\begin{aligned} \text{Ext}_R^i(F \otimes_R C, M^*) &\cong \text{H}_{-i}(\text{Hom}_R(F \otimes_R C, P^*)) \\ &\cong \text{H}_{-i}(\text{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z})). \end{aligned}$$

This is 0 when  $i \neq 0$  because  $\text{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z})$  is a module. Hence, the desired conclusion.

(d) Since  $M$  is  $C$ -injective, it is in  $\mathcal{A}_C(R)$  by Fact 2.7, and so  $\text{Tor}_{\geq 1}^R(C, M) = 0$ . Hence  $M$  is  $C$ -cotorsion by part (c), and it is  $C$ -flat by part (a).  $\square$

**Lemma 3.2.** *Let  $M$  be an  $R$ -module, and let  $C$  be a semidualizing  $R$ -module.*

- (a) *There is an equality  $\mathcal{I}_C\text{-id}_R(M^*) = \mathcal{F}_C\text{-pd}_R(M)$ .*
- (b) *There is an equality  $\mathcal{F}_C\text{-pd}_R(M^*) = \mathcal{I}_C\text{-id}_R(M)$ .*

*Proof.* We prove part (a); the proof of part (b) is similar.

For the inequality  $\mathcal{I}_C\text{-id}_R(M^*) \leq \mathcal{F}_C\text{-pd}_R(M)$ , assume that  $\mathcal{F}_C\text{-pd}_R(M) < \infty$ . Let  $X$  be a  $\mathcal{F}_C(R)$ -resolution of  $M$  such that  $X_i = 0$  for all  $i > \mathcal{F}_C\text{-pd}_R(M)$ . It follows from Lemma 3.1(b) that the complex  $X^*$  is an  $\mathcal{I}_C$ -coresolution of  $M^*$  such that  $X_i^* = 0$  for all  $i > \mathcal{F}_C\text{-pd}_R(M)$ . The desired inequality now follows.

For the reverse inequality, assume that  $j = \mathcal{I}_C\text{-id}_R(M^*) < \infty$ . Fact 2.7 implies that  $M^*$  is in  $\mathcal{A}_C(R)$ , and hence also implies that  $M \in \mathcal{B}_C(R)$ . This condition implies that  $M$  has a  $\mathcal{P}_C$ -resolution  $Z$  by Fact 2.7. In particular, this is an  $\mathcal{F}_C$ -resolution of  $M$ , and so Lemma 3.1(b) implies that  $Z^*$  is an  $\mathcal{I}_C$ -coresolution of  $M^*$ . From [24, (3.3.b)] we know that  $\text{Ker}((\partial_{j+1}^Z)^*) \cong \text{Coker}(\partial_{j+1}^Z)^*$  is in  $\mathcal{I}_C(R)$ . Lemma 3.1(b) implies  $\text{Coker}(\partial_{j+1}^Z) \in \mathcal{F}_C(R)$ . It follows that the truncated complex

$$Z' : 0 \rightarrow \text{Coker}(\partial_{j+1}^Z) \rightarrow Z_{j-1} \rightarrow \cdots \rightarrow Z_0 \rightarrow 0$$

is an  $\mathcal{F}_C$ -resolution of  $M$  such that  $Z'_i = 0$  for all  $i > j$ . The desired inequality now follows, and hence the equality.  $\square$

The next three lemmata document properties of  $\mathcal{F}_C^{\text{cot}}(R)$  for use in the sequel. The first of these contains the characterization of  $C$ -flat  $C$ -cotorsion modules mentioned in Remark 2.5.

**Lemma 3.3.** *Let  $C$  and  $M$  be  $R$ -modules with  $C$  semidualizing. The following conditions are equivalent:*



- (i)  $M \in \mathcal{F}_C^{\text{cot}}(R)$ ;
- (ii)  $M \in \mathcal{F}_C(R)$  and  $\mathcal{F}_C(R) \perp M$ ;
- (iii)  $M \in \mathcal{B}_C(R)$  and  $\text{Hom}_R(C, M) \in \mathcal{F}^{\text{cot}}(R)$ ;
- (iv)  $\text{Hom}_R(C, M) \in \mathcal{F}^{\text{cot}}(R)$ .

In particular, we have  $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ .

*Proof.* (i)  $\iff$  (ii). It suffices to show, for each flat  $R$ -module  $F$ , that  $\mathcal{F}_C(R) \perp F$  if and only if  $\mathcal{F}_C(R) \perp F \otimes_R C$ . Let  $F'$  be a flat  $R$ -module. It suffices to show that

$$\text{Ext}_R^i(F' \otimes_R C, F \otimes_R C) \cong \text{Ext}_R^i(F', F)$$

for each  $i$ . From [26, (1.11.a)] we have the first isomorphism in the next sequence

$$\text{Ext}_R^i(C, F \otimes_R C) \cong \text{Ext}_R^i(C, C) \otimes_R F \cong \begin{cases} R \otimes_R F \cong F & \text{if } i \neq 0 \\ 0 \otimes_R F \cong 0 & \text{if } i = 0 \end{cases}$$

and the second isomorphism is from the fact that  $C$  is semidualizing. Let  $P$  be a projective resolution of  $C$ . The previous display provides a quasiisomorphism

$$\text{Hom}_R(P, F \otimes_R C) \simeq F.$$

Let  $P'$  be a projective resolution of  $F'$ . Hom-tensor adjointness yields the first quasiisomorphism in the next sequence

$$\text{Hom}_R(P' \otimes_R P, F \otimes_R C) \simeq \text{Hom}_R(P', \text{Hom}_R(P, F \otimes_R C)) \simeq \text{Hom}_R(P', F)$$

and the second quasiisomorphism is from the previous display, because  $P'$  is a bounded below complex of projective  $R$ -modules. Since  $F'$  is flat, we conclude that  $P' \otimes_R P$  is a projective resolution of  $F' \otimes_R C$ . It follows that we have

$$\begin{aligned} \text{Ext}_R^i(F' \otimes_R C, F \otimes_R C) &\cong \text{H}_{-i}(\text{Hom}_R(P' \otimes_R P, F \otimes_R C)) \\ &\cong \text{H}_{-i}(\text{Hom}_R(P', F)) \\ &\cong \text{Ext}_R^i(F', F) \end{aligned}$$

as desired.

(i)  $\implies$  (iii). Assume that  $M \in \mathcal{F}_C^{\text{cot}}(R)$ , that is, that  $M \cong C \otimes_R F$  for some  $F \in \mathcal{F}^{\text{cot}}(R) \subseteq \mathcal{A}_C(R)$ . Then

$$\text{Hom}_R(C, M) \cong \text{Hom}_R(C, C \otimes_R F) \cong F \in \mathcal{F}_C^{\text{cot}}(R)$$

and  $M \in \mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$ .

(iii)  $\implies$  (i). If  $M \in \mathcal{B}_C(R)$  and  $\text{Hom}_R(C, M) \in \mathcal{F}^{\text{cot}}(R)$ , then there is an isomorphism  $M \cong C \otimes_R \text{Hom}_R(C, M) \in \mathcal{F}_C^{\text{cot}}(R)$ .

(iii)  $\iff$  (iv). This is from Fact 2.7 because  $\mathcal{F}^{\text{cot}}(R) \subseteq \mathcal{A}_C(R)$ .

The conclusion  $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$  follows from the implication (i)  $\implies$  (ii).  $\square$

**Lemma 3.4.** *If  $C$  is a semidualizing  $R$ -module, then the category  $\mathcal{F}_C^{\text{cot}}(R)$  is closed under products, extensions and summands.*

*Proof.* Consider a set  $\{F_\lambda\}_{\lambda \in \Lambda}$  of modules in  $\mathcal{F}_C^{\text{cot}}(R)$ . From [9, (3.2.24)] we have  $\prod_\lambda F_\lambda \in \mathcal{F}^{\text{cot}}(R)$  and so  $C \otimes_R (\prod_\lambda F_\lambda) \in \mathcal{F}_C^{\text{cot}}(R)$ . Hence, we have

$$\prod_\lambda (C \otimes_R F_\lambda) \cong C \otimes_R (\prod_\lambda F_\lambda) \in \mathcal{F}_C^{\text{cot}}(R)$$

where the isomorphism comes from the fact that  $C$  is finitely presented. Thus  $\mathcal{F}_C^{\text{cot}}(R)$  is closed under products.

By Lemma 1.7(b), the category of  $C$ -cotorsion  $R$ -modules is closed under extensions, and it is closed under summands by the additivity of  $\text{Ext}$ . The category

$\mathcal{F}_C(R)$  is closed under extensions and summands by [18, Props. 5.1(a) and 5.2(a)]. The result now follows from Lemma 3.3.  $\square$

Note that hypotheses of the next lemma are satisfied when  $M \in \mathcal{F}_C(R)^\perp \cap \mathcal{B}_C(R)$ .

**Lemma 3.5.** *Let  $C$  be a semidualizing  $R$ -module, and let  $M$  be a  $C$ -cotorsion  $R$ -module such that the natural evaluation map  $C \otimes_R \text{Hom}_R(C, M) \rightarrow M$  is bijective.*

- (a) *The module  $M$  has an  $\mathcal{F}_C^{\text{cot}}$ -cover, and every  $C$ -flat cover of  $M$  is an  $\mathcal{F}_C^{\text{cot}}$ -cover of  $M$  with  $C$ -cotorsion kernel.*
- (b) *Each  $\mathcal{F}_C^{\text{cot}}$ -precover of  $M$  is surjective.*
- (c) *Assume further that  $\text{Tor}_{\geq 1}^R(C, \text{Hom}_R(C, M)) = 0$ . Then  $M$  has an  $\mathcal{F}_C$ -proper  $\mathcal{F}_C^{\text{cot}}$ -resolution such that  $\text{Ker}(\partial_{i-1}^X)$  is  $C$ -cotorsion for each  $i$ .*

*Proof.* (a) The module  $M$  has a  $C$ -flat cover  $\varphi: F \otimes_R C \rightarrow M$  by [18, Prop. 5.3.a], and  $\text{Ker}(\varphi)$  is  $C$ -cotorsion by [27, (2.1.1)]. Furthermore, the bijectivity of the evaluation map  $C \otimes_R \text{Hom}_R(C, M) \rightarrow M$  implies that there is a projective  $R$ -module  $P$  and a surjective map  $\varphi': P \otimes_R C \twoheadrightarrow M$  by [24, (2.2.a)]. The fact that  $\varphi$  is a precover provides a map  $f: P \otimes_R C \rightarrow F \otimes_R C$  such that  $\varphi' = \varphi f$ . Hence, the surjectivity of  $\varphi'$  implies that  $\varphi$  is surjective. It follows from Lemma 1.7(a) that  $F \otimes_R C$  is  $C$ -cotorsion, and so  $F \otimes_R C \in \mathcal{F}_C^{\text{cot}}(R)$  by Lemma 3.3. Since  $\varphi$  is a  $C$ -flat cover and  $\mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{F}_C(R)$ , we conclude that  $\varphi$  is an  $\mathcal{F}_C^{\text{cot}}$ -cover.

(b) This follows as in part (a) because  $M$  has a surjective  $\mathcal{F}_C^{\text{cot}}$ -cover.

(c) Using parts (a) and (b), the argument of [18, Thm. 2] shows how to construct a resolution with the desired properties.  $\square$

The final three results of this section contain our main conclusions for  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$ . The first of these extends Lemma 3.3.

**Proposition 3.6.** *Let  $C$  and  $M$  be  $R$ -modules with  $C$  semidualizing, and let  $n \geq 0$ . The following conditions are equivalent:*

- (i)  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) \leq n$ ;
- (ii)  $M \in \mathcal{B}_C(R)$  and  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(\text{Hom}_R(C, M)) \leq n$ ;
- (iii)  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(\text{Hom}_R(C, M)) \leq n$ ;
- (iv)  $M \cong C \otimes_R K$  for some  $R$ -module  $K$  such that  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(K) \leq n$ ;
- (v)  $\mathcal{F}_C\text{-pd}_R(M) \leq n$  and  $\mathcal{F}_C(R) \perp M$ .

*Proof.* (i)  $\implies$  (ii) Since  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) \leq n < \infty$ , we have  $M \in \mathcal{B}_C(R)$  by Fact 2.7. Let  $X$  be an  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $M$  such that  $X_i = 0$  when  $i > n$ . For each  $i$ , let  $F_i \in \mathcal{F}_C^{\text{cot}}(R)$  such that  $X_i \cong F_i \otimes_R C$ . Since each  $F_i$  is in  $\mathcal{A}_C(R)$ , we have

$$\text{Hom}_R(C, X)_i \cong \text{Hom}_R(C, X_i) \cong \text{Hom}_R(C, F_i \otimes_R C) \cong F_i.$$

A standard argument using the conditions  $M, X_i \in \mathcal{B}_C(R)$  shows that  $\text{Hom}_R(C, X)$  is an  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $\text{Hom}_R(C, M)$  such that  $\text{Hom}_R(C, X)_i = 0$  when  $i > n$ . The inequality  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(\text{Hom}_R(C, M)) \leq n$  then follows.

(ii)  $\implies$  (iv) The condition  $M \in \mathcal{B}_C(R)$  implies  $M \cong C \otimes_R \text{Hom}_R(C, M)$ , and so  $K = \text{Hom}_R(C, M)$  satisfies the desired conclusions.

(iv)  $\implies$  (v) Let  $F$  be an  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $K$  such that  $F_i = 0$  when  $i > n$ . Using the condition  $K, F_i \in \mathcal{A}_C(R)$ , a standard argument shows that  $C \otimes_R F$  is an  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $C \otimes_R K \cong M$ . Hence, this resolution yields  $\mathcal{F}_C\text{-pd}_R(M) \leq \mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) \leq n$ . By Lemma 3.3, we have  $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ , and so Lemma 1.8 implies  $\mathcal{F}_C(R) \perp \text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$ ; in particular  $\mathcal{F}_C(R) \perp M$ .

(v)  $\implies$  (i) The assumption  $\mathcal{F}_C\text{-pd}_R(M) \leq n$  implies  $M \in \mathcal{B}_C(R)$  by Fact 2.7, and so  $\text{Ext}_R^{\geq 1}(C, M) = 0$ . Lemma 3.5(c) implies that  $M$  has an  $\mathcal{F}_C$ -proper  $\mathcal{F}_C^{\text{cot}}$ -resolution  $X$  such that  $K_i = \text{Ker}(\partial_{i-1}^X)$  is  $C$ -cotorsion for each  $i$ . In particular, the truncated complex

$$X' = 0 \rightarrow K_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$$

is exact and  $\text{Hom}_R(C, -)$ -exact. Since  $\mathcal{F}_C\text{-pd}_R(M) \leq n$ , the proof of the implication (i)  $\implies$  (ii) shows that  $\text{fd}_R(\text{Hom}_R(C, M)) \leq n$ . Since each  $R$ -module  $\text{Hom}_R(C, X_i)$  is flat by Lemma 3.3, the exact complex  $\text{Hom}_R(C, X')$  is a truncation of an augmented flat resolution of  $\text{Hom}_R(C, M)$ . It follows that  $\text{Hom}_R(C, K_n)$  is flat, and so  $K_n \in \mathcal{F}_C(R)$  by [18, Thm. 1]. Hence  $X'$  is an augmented  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $M$ , and so  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) \leq n$ .

(ii)  $\iff$  (iii) follows from Fact 2.7 because  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)} \subseteq \mathcal{A}_C(R)$ .  $\square$

**Lemma 3.7.** *Let  $C$  be a semidualizing  $R$ -module. If  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) < \infty$ , then any bounded  $\mathcal{F}_C^{\text{cot}}$ -resolution  $X$  of  $M$  is  $\mathcal{F}_C$ -proper.*

*Proof.* Observe that  $\mathcal{F}_C(R) \perp X_i$  for all  $i$  and  $\mathcal{F}_C(R) \perp M$  by Proposition 3.6. So, the complex  $X^+$  is exact and such that  $(X^+)_i = 0$  for  $i \gg 0$  and  $\mathcal{F}_C(R) \perp (X^+)_i$ . Hence, Lemma 1.9(b) implies that  $X^+$  is  $\text{Hom}_R(\mathcal{F}_C, -)$ -exact.  $\square$

**Proposition 3.8.** *Let  $C$  be a semidualizing  $R$ -module. The category  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$  is closed under extensions, cokernels of monomorphisms and summands.*

*Proof.* Consider an exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

such that  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_1)$  and  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_3)$  are finite. To show that  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$  is closed under extensions we need to show that  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_2)$  is finite.

The condition  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_1) < \infty$  implies  $\mathcal{I}_C\text{-id}(M_1^*) = \mathcal{F}_C\text{-pd}_R(M_1) < \infty$  by Lemma 3.2(a) and Proposition 3.6; and similarly  $\mathcal{I}_C\text{-id}(M_3^*) < \infty$ . From [24, (3.4)] we know that the category of  $R$ -modules of finite  $\mathcal{I}_C$ -injective dimension is closed under extensions. Using the dual exact sequence

$$0 \rightarrow M_3^* \rightarrow M_2^* \rightarrow M_1^* \rightarrow 0$$

we conclude that  $\mathcal{I}_C\text{-id}(M_2^*)$  is finite. Thus, Lemma 3.2(a) implies that  $\mathcal{F}_C\text{-pd}_R(M_2)$  is finite.

Since  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_1) < \infty$ , Proposition 3.6 implies  $\mathcal{F}_C(R) \perp M_1$ ; and similarly  $\mathcal{F}_C(R) \perp M_3$ . Thus, we have  $\mathcal{F}_C(R) \perp M_2$  by Lemma 1.7(b). Combining this with the previous paragraph, Proposition 3.6 implies that  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M_2) < \infty$ .

The proof of the fact that  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$  is closed under cokernels of monomorphisms is similar. The fact that  $\text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$  is closed under summands is even easier to prove using the natural isomorphism  $(M_1 \oplus M_2)^* \cong M_1^* \oplus M_2^*$ .  $\square$

#### 4. WEAK AB-CONTEXT

Let  $C$  be a semidualizing  $R$ -module. The point of this section is to show that the triple  $(\mathcal{GF}_C(R), \text{res } \widehat{\mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$  is a weak AB-context, and to document the immediate consequences; see Theorem I and Corollary 4.10. We begin the section with two results modeled on [16, (3.22) and (3.6)].

**Lemma 4.1.** *If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{GF}_C(R) \perp \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ .*

*Proof.* By Lemma 1.8 it suffices to show  $\mathcal{GF}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ . Fix modules  $M \in \mathcal{GF}_C(R)$  and  $N \in \mathcal{F}_C^{\text{cot}}(R)$ . By Lemma 3.1, we know that the Pontryagin dual  $N^*$  is  $C$ -injective. Hence, for  $i \geq 1$ , the vanishing in the next sequence is from Fact 2.9

$$\text{Ext}_R^i(M, N^{**}) \cong \text{Ext}_R^i(M, \text{Hom}_{\mathbb{Z}}(N^*, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_R^i(M, N^*), \mathbb{Q}/\mathbb{Z}) = 0.$$

The second isomorphism is a form of Hom-tensor adjointness using the fact that  $\mathbb{Q}/\mathbb{Z}$  is injective over  $\mathbb{Z}$ . To finish the proof, it suffices to show that  $N$  is a summand of  $N^{**}$ ; then the last sequence shows  $\text{Ext}_R^{\geq 1}(M, N) = 0$ . Write  $N \cong C \otimes_R F$  for some flat cotorsion  $R$ -module  $F$ , and use Hom-tensor adjointness to conclude

$$N^* \cong \text{Hom}_{\mathbb{Z}}(C \otimes_R F, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_R(C, \text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})).$$

Lemma 2.3(b) implies that  $\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$  is injective, so the proof of Lemma 3.1(a) explains the second isomorphism in the next sequence

$$N^{**} \cong \text{Hom}_R(C, \text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}))^* \cong C \otimes_R \text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong C \otimes_R F^{**}.$$

The proof of [16, (3.22)] shows that  $F$  is a summand of  $F^{**}$ , and it follows that  $N \cong C \otimes_R F$  is a summand of  $C \otimes_R F^{**} \cong N^{**}$ , as desired.  $\square$

**Lemma 4.2.** *Let  $C$  be a semidualizing  $R$ -module. If  $M$  is an  $R$ -module, then  $M$  is in  $\mathcal{GF}_C(R)$  if and only if its Pontryagin dual  $M^*$  is in  $\mathcal{GI}_C(R)$ .*

*Proof.* Consider the trivial extension  $R \ltimes C$  from Fact 2.9. By [16, (3.6)] we know that  $M$  is in  $\mathcal{GF}(R \ltimes C)$  if and only if  $M^*$  is in  $\mathcal{GI}(R \ltimes C)$ . Also  $M$  is in  $\mathcal{GF}(R \ltimes C)$  if and only if  $M$  is in  $\mathcal{GF}_C(R)$ , and  $M^*$  is in  $\mathcal{GI}(R \ltimes C)$  if and only if  $M^*$  is in  $\mathcal{GI}_C(R)$  by Fact 2.9. Hence, the equivalence.  $\square$

The following result establishes Theorem I(a).

**Proposition 4.3.** *Let  $C$  be a semidualizing  $R$ -module. The category  $\mathcal{GF}_C(R)$  is closed under kernels of epimorphisms, extensions and summands.*

*Proof.* The result dual to [26, (3.8)] says that  $\mathcal{GI}_C(R)$  is closed under cokernels of monomorphisms, extensions and summands. To see that  $\mathcal{GF}_C(R)$  is closed under summands, let  $M \in \mathcal{GF}_C(R)$  and assume that  $N$  is a direct summand of  $M$ . It follows that the Pontryagin dual  $N^*$  is a direct summand of  $M^*$ . Lemma 4.2 implies that  $M^*$  is in  $\mathcal{GI}_C(R)$  which is closed under summands. We conclude that  $N^* \in \mathcal{GI}_C(R)$ , and so  $N \in \mathcal{GF}_C(R)$ . Hence  $\mathcal{GF}_C(R)$  is closed under summands, and the other properties are verified similarly.  $\square$

The next four results put the finishing touches on Theorem I.

**Lemma 4.4.** *Let  $C$  be a semidualizing  $R$ -module. If  $X$  is a complete  $\mathcal{FF}_C$ -resolution, then  $\text{Coker}(\partial_n^X) \in \mathcal{GF}_C(R)$  for each  $n \in \mathbb{Z}$ .*

*Proof.* Write  $M_n = \text{Coker}(\partial_n^X)$ , and note that  $M_1 \in \mathcal{GF}_C(R)$  by definition. Fact 2.9 implies that  $X_n \in \mathcal{GF}_C(R)$  for each  $n \in \mathbb{Z}$ . Since  $M_1$  is in  $\mathcal{GF}_C(R)$ , an induction argument using Proposition 4.3 shows  $M_n \in \mathcal{GF}_C(R)$  for each  $n \geq 1$ .

Now assume  $n \leq 0$ . Lemma 1.9(c), implies  $\text{Tor}_{\geq 1}^R(M_n, \mathcal{I}_C) = 0$ . By construction, the following sequence is exact and  $- \otimes_R \mathcal{I}_C$ -exact

$$0 \rightarrow M_n \rightarrow X_{n-2} \rightarrow X_{n-3} \cdots$$

with each  $X_{n-i} \in \mathcal{GF}_C(R)$ , and so  $M_n \in \mathcal{GF}_C(R)$  by Fact 2.9.  $\square$

**Lemma 4.5.** *Let  $C$  be a semidualizing  $R$ -module. If  $M \in \mathcal{F}_C(R)$ , then there is an exact sequence  $0 \rightarrow M \rightarrow M_1 \rightarrow M_2 \rightarrow 0$  with  $M_1 \in \mathcal{F}_C^{\text{cot}}(R)$  and  $M_2 \in \mathcal{F}_C(R)$ .*

*Proof.* Since  $M$  is  $C$ -flat, we know from [18, Thm. 1] that  $\text{Hom}_R(C, M)$  is flat. By [27, (3.1.6)] there is a cotorsion flat module  $F$  containing  $\text{Hom}_R(C, M)$  such that the quotient  $F/\text{Hom}_R(C, M)$  is flat. Consider the exact sequence

$$0 \rightarrow \text{Hom}_R(C, M) \rightarrow F \rightarrow F/\text{Hom}_R(C, M) \rightarrow 0.$$

Since  $F/\text{Hom}_R(C, M)$  is flat, an application of  $C \otimes_R -$  yields an exact sequence

$$0 \rightarrow C \otimes_R \text{Hom}_R(C, M) \rightarrow C \otimes_R F \rightarrow C \otimes_R (F/\text{Hom}_R(C, M)) \rightarrow 0.$$

Because  $M$  is  $C$ -flat, it is in  $\mathcal{B}_C(R)$  and so  $C \otimes_R \text{Hom}_R(C, M) \cong M$ . With  $M_1 = C \otimes_R F$  and  $M_2 = C \otimes_R (F/\text{Hom}_R(C, M))$  this yields the desired sequence.  $\square$

**Lemma 4.6.** *Let  $C$  be a semidualizing  $R$ -module. Each module  $M \in \mathcal{GF}_C(R)$  admits an injective  $\mathcal{F}_C^{\text{cot}}$ -preenvelope  $\alpha: M \rightarrow Y$  such that  $\text{Coker}(\alpha) \in \mathcal{GF}_C(R)$ .*

*Proof.* Let  $M \in \mathcal{GF}_C(R)$  with complete  $\mathcal{FF}_C$ -resolution  $X$ . By definition, this says that  $M$  is a submodule of the  $C$ -flat  $R$ -module  $X_{-1}$ , and Lemma 4.4 implies that  $X_{-1}/M \in \mathcal{GF}_C(R)$ . Since  $X_{-1}$  is  $C$ -flat, Lemma 4.5 yields an exact sequence

$$0 \rightarrow X_{-1} \rightarrow Z \rightarrow Z/X_{-1} \rightarrow 0$$

with  $Z \in \mathcal{F}_C^{\text{cot}}(R)$  and  $Z/X_{-1} \in \mathcal{F}_C(R)$ . It follows that  $Z/X_{-1}$  is in  $\mathcal{GF}_C(R)$ . Since  $X_{-1}/M$  is also in  $\mathcal{GF}_C(R)$ , and  $\mathcal{GF}_C(R)$  is closed under extensions by Proposition 4.3, the following exact sequence shows that  $Z/M$  is also in  $\mathcal{GF}_C(R)$

$$0 \rightarrow X_{-1}/M \rightarrow Z/M \rightarrow Z/X_{-1} \rightarrow 0.$$

In particular, Lemma 4.1 implies  $Z/M \perp \mathcal{F}_C^{\text{cot}}(R)$ , and it follows that the next sequence is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact by Lemma 1.7(a).

$$0 \rightarrow M \rightarrow C \otimes_R F \rightarrow Z/M \rightarrow 0$$

The conditions  $Z \in \mathcal{F}_C^{\text{cot}}(R)$  and  $Z/M \in \mathcal{GF}_C(R)$  then implies that the inclusion  $M \rightarrow Z$  is an  $\mathcal{F}_C^{\text{cot}}$ -preenvelope whose cokernel is in  $\mathcal{GF}_C(R)$ .  $\square$

**Proposition 4.7.** *Let  $C$  be a semidualizing  $R$ -module. The category  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for the category  $\mathcal{GF}_C(R)$ . In particular, every module in  $\mathcal{GF}_C(R)$  admits a  $\mathcal{F}_C^{\text{cot}}$ -proper  $\mathcal{F}_C^{\text{cot}}$ -coresolution, and so  $\mathcal{GF}_C(R) \subseteq \text{cores } \widehat{\mathcal{F}_C^{\text{cot}}}(R)$ .*

*Proof.* Lemmas 4.1 and 4.6 imply that  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for  $\mathcal{GF}_C(R)$ . The remaining conclusions follow immediately.  $\square$

**Lemma 4.8.** *If  $C$  is a semidualizing  $R$ -module, then there is an equality  $\mathcal{F}_C^{\text{cot}}(R) = \mathcal{GF}_C(R) \cap \text{res } \widehat{\mathcal{F}_C^{\text{cot}}}(R)$ .*

*Proof.* The containment  $\mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{GF}_C(R) \cap \text{res } \widehat{\mathcal{F}_C^{\text{cot}}}(R)$  is straightforward; see Definition 1.4 and Fact 2.9. For the reverse containment, let  $M \in \mathcal{GF}_C(R) \cap \text{res } \widehat{\mathcal{F}_C^{\text{cot}}}(R)$ . Truncate a bounded  $\mathcal{F}_C^{\text{cot}}$ -resolution to obtain an exact sequence

$$0 \rightarrow K \rightarrow F \otimes_R C \rightarrow M \rightarrow 0$$

with  $F \in \mathcal{F}_C^{\text{cot}}(R)$  and such that  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(K) < \infty$ . We have  $\text{Ext}_R^1(M, K) = 0$  by Lemma 4.1, so this sequence splits. Hence  $M$  is a summand of  $F \otimes_R C \in \mathcal{F}_C^{\text{cot}}(R)$ . Lemma 3.4 implies that  $\mathcal{F}_C^{\text{cot}}(R)$  is closed under summands, so  $M \in \mathcal{F}_C^{\text{cot}}(R)$ .  $\square$

**4.9. Proof of Theorem I.** Part (a) is in Proposition 4.3. Since  $\mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{GF}_C(R)$  by Fact 2.9, we have  $\widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)} \subseteq \widehat{\text{res } \mathcal{GF}_C(R)}$ . With this, part (b) follows from Proposition 3.8. Proposition 4.7 and Lemma 4.8 justify part (c).  $\square$

Here is the list of immediate consequences of Theorem I and [15, (1.12.10)]. For part (a), recall that  $\text{add}(\mathcal{X})$  is the subcategory of all  $R$ -modules isomorphic to a direct summand of a finite direct sum of modules in  $\mathcal{X}$ .

**Corollary 4.10.** *Let  $C$  be a semidualizing  $R$ -module and let  $M \in \widehat{\text{res } \mathcal{GF}_C(R)}$ .*

- (a) *If  $\mathcal{X}$  is an injective cogenerator for  $\mathcal{GF}_C(R)$ , then  $\text{add}(\mathcal{X}) = \mathcal{F}_C^{\text{cot}}(R)$ .*
- (b) *There exists an exact sequence  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  with  $X \in \mathcal{GF}_C(R)$  and  $Y \in \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ .*
- (c) *There exists an exact sequence  $0 \rightarrow M \rightarrow Y \rightarrow X \rightarrow 0$  with  $X \in \mathcal{GF}_C(R)$  and  $Y \in \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ .*
- (d) *The following conditions are equivalent:*
  - (i)  $M \in \mathcal{GF}_C(R)$ ;
  - (ii)  $\text{Ext}_R^{\geq 1}(M, \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}) = 0$ ;
  - (iii)  $\text{Ext}_R^1(M, \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}) = 0$ ;
  - (iv)  $\text{Ext}_R^{\geq 1}(M, \mathcal{F}_C^{\text{cot}}) = 0$ .

*Thus, the surjection  $X \rightarrow M$  from (b) is a  $\mathcal{GF}_C$ -precover of  $M$ .*

- (e) *The following conditions are equivalent:*
  - (i)  $M \in \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}$ ;
  - (ii)  $\text{Ext}_R^{\geq 1}(\mathcal{GF}_C, M) = 0$ ;
  - (iii)  $\text{Ext}_R^1(\mathcal{GF}_C, M) = 0$ ;
  - (iv)  $\sup\{i \geq 0 \mid \text{Ext}_R^i(\mathcal{GF}_C, M) \neq 0\} < \infty$  and  $\text{Ext}_R^{\geq 1}(\mathcal{F}_C^{\text{cot}}, M) = 0$ .

*Thus, the injection  $M \rightarrow Y$  from (c) is a  $\widehat{\text{res } \mathcal{F}_C^{\text{cot}}}$ -preenvelope of  $M$ .*

- (f) *There are equalities*

$$\begin{aligned} \mathcal{GF}_C\text{-pd}_R(M) &= \sup\{i \geq 0 \mid \text{Ext}_R^i(M, \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}) \neq 0\} \\ &= \sup\{i \geq 0 \mid \text{Ext}_R^i(M, \mathcal{F}_C^{\text{cot}}) \neq 0\} \end{aligned}$$

- (g) *There is an inequality  $\mathcal{GF}_C\text{-pd}_R(M) \leq \mathcal{F}_C^{\text{cot}}\text{-pd}_R(M)$  with equality when  $\mathcal{F}_C^{\text{cot}}\text{-pd}_R(M) < \infty$ .*
- (h) *The category  $\widehat{\text{res } \mathcal{GF}_C(R)}$  is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.*  $\square$

For the next result recall that the triple  $(\mathcal{GF}_C(R), \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$  is an AB-context if it is a weak AB-context and such that  $\widehat{\text{res } \mathcal{GF}_C(R)} = \mathcal{M}(R)$ .

**Proposition 4.11.** *Assume that  $\dim(R)$  is finite, and let  $C$  be a semidualizing  $R$ -module. The triple  $(\mathcal{GF}_C(R), \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$  is an AB-context if and only if  $C$  is dualizing for  $R$ .*

*Proof.* Assume first that  $(\mathcal{GF}_C(R), \widehat{\text{res } \mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$  is an AB-context. Recall that every maximal ideal of the trivial extension  $R \ltimes C$  is of the form  $\mathfrak{m} \ltimes C$  for some maximal ideal  $\mathfrak{m} \subset R$ , and there is an isomorphism  $(R \ltimes C)/(\mathfrak{m} \ltimes C) \cong R/\mathfrak{m}$ .

With Fact 2.9, this yields the equality in the next sequence

$$\begin{aligned} \text{Gfd}_{(R \ltimes C)_{\mathfrak{m} \ltimes C}}((R \ltimes C)_{\mathfrak{m} \ltimes C}/(\mathfrak{m} \ltimes C)_{\mathfrak{m} \ltimes C}) &\leq \text{Gfd}_{R \ltimes C}((R \ltimes C)/(\mathfrak{m} \ltimes C)) \\ &= \mathcal{GF}_C\text{-pd}_R(R/\mathfrak{m}) < \infty. \end{aligned}$$

The first inequality follows from [5, (5.1.3)], and the finiteness is by assumption. Using [5, (1.2.7),(1.4.9),(5.1.11)] we deduce that the following ring is Gorenstein

$$(R \ltimes C)_{\mathfrak{m} \ltimes C} \cong R_{\mathfrak{m}} \ltimes C_{\mathfrak{m}}$$

and so [21, (7)] implies that  $C_{\mathfrak{m}}$  is dualizing for  $R_{\mathfrak{m}}$ . (This also follows from [6, (8.1)] and [17, (3.1)].) Since this is true for each maximal ideal of  $R$  and  $\dim(R) < \infty$ , we conclude that  $C$  is dualizing for  $R$  by [14, (5.8.2)].

Conversely, assume that  $C$  is dualizing for  $R$ . Using Theorem I, it suffices to show that each  $R$ -module  $M$  has  $\mathcal{GF}_C\text{-pd}_R(M) < \infty$ . Since  $C$  is dualizing, the trivial extension  $R \ltimes C$  is Gorenstein by [21, (7)]. Also, we have  $\dim(R \ltimes C) = \dim(R) < \infty$  as  $\text{Spec}(R \ltimes C)$  is in bijection with  $\text{Spec}(R)$ . Thus, in the next sequence

$$\mathcal{GF}_C\text{-pd}_R(M) = \text{Gfd}_{R \ltimes C}(M) < \infty$$

the finiteness is from [9, (12.3.1)] and the equality is from Fact 2.9.  $\square$

To end this section, we prove a compliment to [26, (4.6)] which establishes the existence of certain approximations. For this, we need the following preliminary result which compares to Lemma 4.8.

**Lemma 4.12.** *If  $C$  is a semidualizing  $R$ -module, then there is an equality  $\mathcal{F}_C(R) = \mathcal{GF}_C(R) \cap \widehat{\text{res } \mathcal{F}_C(R)}$ .*

*Proof.* The containment  $\mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R) \cap \widehat{\text{res } \mathcal{F}_C(R)}$  is from Definition 1.4 and Fact 2.9. For the reverse containment, let  $M \in \mathcal{GF}_C(R) \cap \widehat{\text{res } \mathcal{F}_C(R)}$ . Let  $n \geq 1$  be an integer with  $\mathcal{F}_C\text{-pd}_R(M) \leq n$ . We show by induction on  $n$  that  $M$  is  $C$ -flat.

For the base case  $n = 1$ , there is an exact sequence

$$(\dagger) \quad 0 \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

with  $X_1, X_0 \in \mathcal{F}_C(R)$ . Lemma 4.5 provides an exact sequence

$$(\ddagger) \quad 0 \rightarrow X_1 \rightarrow Y_1 \rightarrow Y_2 \rightarrow 0$$

with  $Y_1 \in \mathcal{F}_C^{\text{cot}}(R)$  and  $Y_2 \in \mathcal{F}_C(R)$ . Consider the following pushout diagram whose top row is  $(\dagger)$  and whose leftmost column is  $(\ddagger)$ .

$$(*) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \cong \downarrow \\ 0 & \longrightarrow & Y_1 & \longrightarrow & V & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & Y_2 & \xrightarrow{\cong} & Y_2 & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since  $M$  is in  $\mathcal{GF}_C(R)$  and  $Y_1$  is in  $\mathcal{F}_C^{\text{cot}}(R)$ , Lemma 4.1 implies  $\text{Ext}_R^1(M, Y_1) = 0$ . Hence, the middle row of  $(*)$  splits. The subcategory  $\mathcal{F}_C(R)$  is closed under extensions and summands by [18, Props. 5.1(a) and 5.2(a)]. Hence, the middle column of  $(*)$  shows that  $V \in \mathcal{F}_C(R)$ , so the fact that the middle row of  $(*)$  splits implies that  $M \in \mathcal{F}_C(R)$ , as desired.

For the induction step, assume that  $n \geq 2$ . Truncate a bounded  $\mathcal{F}_C$ -resolution of  $M$  to find an exact sequence

$$0 \rightarrow K \rightarrow Z \rightarrow M \rightarrow 0$$

such that  $Z \in \mathcal{F}_C(R)$  and  $\mathcal{F}_C\text{-pd}_R(K) \leq n - 1$ . By induction, we conclude that  $K \in \mathcal{F}_C(R)$ . Hence, the displayed sequence implies  $\mathcal{F}_C\text{-pd}_R(M) \leq 1$ , and the base case implies that  $M \in \mathcal{F}_C(R)$ .  $\square$

**Proposition 4.13.** *Let  $C$  be a semidualizing  $R$ -module and assume that  $\dim(R)$  is finite. If  $M \in \mathcal{GF}_C(R)$ , then there exists an exact sequence*

$$0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0$$

*such that  $K \in \mathcal{F}_C(R)$  and  $X \in \mathcal{GP}_C(R)$ .*

*Proof.* Since  $M$  is in  $\mathcal{GF}_C(R)$  and  $\dim(R) < \infty$ , we know that  $\mathcal{GP}_C\text{-pd}_R(M) < \infty$  by [22, (3.3.c)]. Hence, from [26, (4.6)] there is an exact sequence

$$0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0$$

with  $K \in \text{res } \widehat{\mathcal{P}_C(R)}$  and  $X \in \mathcal{GP}_C(R)$ . From [22, (3.3.a)] we have  $X \in \mathcal{GP}_C(R) \subseteq \mathcal{GF}_C(R)$ . Since  $\mathcal{GF}_C(R)$  is closed under kernels of epimorphisms by Proposition 4.3, the displayed sequence implies that  $K \in \mathcal{GF}_C(R)$ . The containment  $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$  implies  $K \in \text{res } \widehat{\mathcal{P}_C(R)} \subseteq \text{res } \widehat{\mathcal{F}_C(R)}$ , and so Lemma 4.12 says  $K \in \mathcal{F}_C(R)$ . Thus, the displayed sequence has the desired properties.  $\square$

## 5. STABILITY OF CATEGORIES

This section contains our analysis of the categories  $\mathcal{G}^n(\mathcal{F}_C(R))$  and  $\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R))$ ; see Definition 2.10. We draw many of our conclusions from the known behavior for  $\mathcal{G}^n(\mathcal{I}_C(R))$  using Pontryagin duals. This requires, however, the use of the categories  $\mathcal{H}_C^n(\mathcal{F}_C(R))$  and  $\mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R))$  as a bridge; see Definition 2.12.

**Lemma 5.1.** *Let  $C$  be a semidualizing  $R$ -module, and let  $X$  be an  $R$ -complex. If  $X$  is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact, then it is  $-\otimes_R \mathcal{I}_C$ -exact.*

*Proof.* Let  $N \in \mathcal{I}_C(R)$ . From Lemmas 3.1(d) and 3.3 we know that the Pontryagin dual  $N^*$  is in  $\mathcal{F}_C^{\text{cot}}(R)$ . Hence, the following complex is exact by assumption

$$\text{Hom}_R(X, N^*) \cong \text{Hom}_R(X, \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(X \otimes_R N, \mathbb{Q}/\mathbb{Z}).$$

As  $\mathbb{Q}/\mathbb{Z}$  is faithfully injective over  $\mathbb{Z}$ , we conclude that  $X \otimes_R N$  is exact, and so  $X$  is  $-\otimes_R \mathcal{I}_C$ -exact.  $\square$

Note that the hypotheses of the next lemma are satisfied whenever  $\mathcal{X} \subseteq \mathcal{GF}_C(R)$  by Fact 2.9 and Lemma 4.1.

**Lemma 5.2.** *Let  $C$  be a semidualizing  $R$ -module and  $\mathcal{X}$  a subcategory of  $\mathcal{M}(R)$ .*

- (a) *If  $\text{Tor}_{\geq 1}^R(\mathcal{X}, \mathcal{I}_C) = 0$ , then  $\text{Tor}_{\geq 1}^R(\mathcal{H}_C^n(\mathcal{X}), \mathcal{I}_C) = 0$  for each  $n \geq 1$ .*
- (b) *If  $\mathcal{X} \perp \mathcal{F}_C^{\text{cot}}(R)$ , then  $\mathcal{H}_C^n(\mathcal{X}) \perp \mathcal{F}_C^{\text{cot}}(R)$  for each  $n \geq 1$ .*



*Proof.* By induction on  $n$ , it suffices to prove the result for  $n = 1$ . We prove part (a). The proof of part (b) is similar.

Let  $M \in \mathcal{H}_C(\mathcal{X})$  with  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{X}$ -resolution  $X$ . The complex  $X$  is  $-\otimes_R \mathcal{I}_C$ -exact by Lemma 5.1. Since we have assumed that  $\text{Tor}_{\geq 1}^R(\mathcal{X}, \mathcal{I}_C) = 0$ , the desired conclusion follows from Lemma 1.9(c) because  $M \cong \text{Ker}(\partial_{-1}^X)$ .  $\square$

The converse of the next result is in Proposition 5.5.

**Lemma 5.3.** *If  $C$  is a semidualizing  $R$ -module and  $M \in \mathcal{H}_C(\mathcal{F}_C(R))$ , then  $M^* \in \mathcal{G}(\mathcal{I}_C(R))$ .*

*Proof.* Let  $X$  be a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C$ -resolution of  $M$ . Lemma 3.1(b) implies that the complex  $X^* = \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$  is an exact complex in  $\mathcal{I}_C(R)$ . Furthermore  $M^* \cong \text{Coker}(\partial_1^{X^*})$ . Thus, it suffices to show that  $X^*$  is  $\text{Hom}_R(\mathcal{I}_C, -)$ -exact and  $\text{Hom}_R(-, \mathcal{I}_C)$ -exact. Let  $I$  be an injective  $R$ -module.

The second isomorphism in the following sequence is Hom-evaluation [7, (0.3.b)]

$$C \otimes_R X^* \cong C \otimes_R \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(C, X), \mathbb{Q}/\mathbb{Z}).$$

Since  $\text{Hom}_R(C, X)$  is exact by assumption, we conclude that  $C \otimes_R X^* \cong X^* \otimes_R C$  is also exact. It follows that the following complexes are also exact

$$\text{Hom}_R(X^* \otimes_R C, I) \cong \text{Hom}_R(X^*, \text{Hom}_R(C, I))$$

where the isomorphism is Hom-tensor adjointness. Thus  $X^*$  is  $\text{Hom}_R(-, \mathcal{I}_C)$ -exact.

Lemma 5.1 implies that the complex  $\text{Hom}_R(C, I) \otimes_R X$  is exact. Hence, the following complexes are also exact

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(C, I) \otimes_R X, \mathbb{Q}/\mathbb{Z}) &\cong \text{Hom}_R(\text{Hom}_R(C, I), \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \\ &\cong \text{Hom}_R(\text{Hom}_R(C, I), X^*) \end{aligned}$$

and so  $X^*$  is  $\text{Hom}_R(\mathcal{I}_C, -)$ -exact.  $\square$

The next result is a version of [23, (5.2)] for  $\mathcal{H}_C(\mathcal{F}_C(R))$ .

**Proposition 5.4.** *If  $C$  is a semidualizing  $R$ -module, then there is an equality  $\mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ .*

*Proof.* For the containment  $\mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ , let  $M \in \mathcal{H}_C(\mathcal{F}_C(R))$ , and let  $X$  be a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C$ -resolution of  $M$ . Lemma 5.1 implies that  $X$  is  $-\otimes_R \mathcal{I}_C$ -exact, and so the sequence

$$0 \rightarrow M \rightarrow X_{-1} \rightarrow X_{-2} \rightarrow \cdots$$

satisfies condition 2.9(1). Fact 2.9 implies  $\text{Tor}_{\geq 1}^R(\mathcal{F}_C, \mathcal{I}_C) = 0$  and so Lemma 5.2(a) provides  $\text{Tor}_{\geq 1}^R(M, \mathcal{I}_C) = 0$ . From Fact 2.9 we conclude  $M \in \mathcal{GF}_C(R)$ . Also, Lemma 5.3 guarantees that  $M^* \in \mathcal{G}(\mathcal{I}_C(R))$ , and so  $M^* \in \mathcal{A}_C(R)$  by Fact 2.11. Thus, Fact 2.7 implies  $M \in \mathcal{B}_C(R)$ .

For the reverse containment, let  $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ , and let  $Y$  be a complete  $\mathcal{FF}_C$ -resolution of  $M$ . In particular, the complex

$$(\dagger) \quad 0 \rightarrow M \rightarrow Y_{-1} \rightarrow Y_{-2} \rightarrow \cdots$$

is an augmented  $\mathcal{F}_C$ -coresolution of  $M$  and is  $-\otimes_R \mathcal{I}_C$ -exact. We claim that this complex is also  $\text{Hom}_R(C, -)$ -exact and  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact. For each  $i \in \mathbb{Z}$  set  $M_i = \text{Coker}(\partial_i^Y)$ . This yields an isomorphism  $M \cong M_1$ . By assumption, we have  $M, Y_i \in \mathcal{B}_C(R)$  for each  $i < 0$ , and so  $C \perp M$  and  $C \perp Y_i$ . Thus, Lemma 1.9(b)

implies that the complex  $(\dagger)$  is  $\text{Hom}_R(C, -)$ -exact. From Lemma 4.4 we conclude  $M_i \in \mathcal{GF}_C(R)$  for each  $i$ , and so  $M_i \perp \mathcal{F}_C^{\text{cot}}(R)$  by Lemma 4.1. Lemma 3.3 implies  $Y_i \perp \mathcal{F}_C^{\text{cot}}(R)$  for each  $i < 0$ , and so Lemma 1.9(a) guarantees that  $(\dagger)$  is also  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact.

Because  $M \in \mathcal{B}_C(R)$ , Fact 2.7 provides an augmented  $\mathcal{P}_C$ -proper  $\mathcal{P}_C$ -resolution

$$(\ddagger) \quad \cdots \xrightarrow{\partial_2^Z} Z_1 \xrightarrow{\partial_1^Z} Z_0 \rightarrow M \rightarrow 0.$$

Since each  $Z_i \in \mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ , we have  $Z_i \perp \mathcal{F}_C^{\text{cot}}(R)$  by Lemma 3.3. Since  $M \perp \mathcal{F}_C^{\text{cot}}(R)$ , we see from Lemma 1.9(a) that  $(\ddagger)$  is also  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact.

It follows that the complex obtained by splicing the sequences  $(\dagger)$  and  $(\ddagger)$  is a  $\mathcal{P}_C \mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C$ -resolution of  $M$ . Thus  $M \in \mathcal{H}_C(\mathcal{F}_C(R))$ , as desired.  $\square$

Our next result contains the converse to Lemma 5.3.

**Proposition 5.5.** *Let  $C$  be a semidualizing  $R$ -module and  $M$  an  $R$ -module. Then  $M \in \mathcal{H}_C(\mathcal{F}_C(R))$  if and only if  $M^* \in \mathcal{G}(\mathcal{I}_C(R))$ .*

*Proof.* One implication is in Lemma 5.3. For the converse, assume that  $M^*$  is in  $\mathcal{G}(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ ; see Fact 2.11. Fact 2.7 and Lemma 4.2 combine with Proposition 5.4 to yield  $M \in \mathcal{B}_C(R) \cap \mathcal{GF}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$ .  $\square$

The next three lemmata are for use in Theorem 5.9.

**Lemma 5.6.** *If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{H}_C^2(\mathcal{F}_C(R)) \subseteq \mathcal{B}_C(R)$ .*

*Proof.* Let  $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$  and let  $X$  be a  $\mathcal{P}_C \mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{H}_C(\mathcal{F}_C)$ -resolution of  $M$ . In particular, the complex  $\text{Hom}_R(C, X)$  is exact. Each module  $X_i$  is in  $\mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{B}_C(R)$  by Proposition 5.4, and so  $\text{Ext}_R^{\geq 1}(C, X_i) = 0$  for each  $i$ . Thus, Lemma 1.9(b) implies that  $\text{Ext}_R^{\geq 1}(C, M) = 0$ . Also, since  $M \cong \text{Ker}(\partial_{-1}^X)$ , the left-exactness of  $\text{Hom}_R(C, -)$  implies that  $\text{Hom}_R(C, M) \cong \text{Ker}(\partial_{-1}^{\text{Hom}_R(C, X)})$ .

The natural evaluation map  $C \otimes_R \text{Hom}_R(C, X_i) \rightarrow X_i$  is an isomorphism for each  $i$  because  $X_i \in \mathcal{B}_C(R)$ , and so we have  $C \otimes_R \text{Hom}_R(C, X) \cong X$ . In particular, the complex  $\text{Hom}_R(C, X)$  is  $- \otimes_R C$ -exact. As  $\text{Tor}_{\geq 1}^R(C, \text{Hom}_R(C, X_i)) = 0$  for each  $i$ , Lemma 1.9(c) implies that  $\text{Tor}_{\geq 1}^R(C, \text{Hom}_R(C, M)) = 0$ .

Finally, each row in the following diagram is exact

$$\begin{array}{ccccccc} C \otimes_R \text{Hom}_R(C, X_1) & \longrightarrow & C \otimes_R \text{Hom}_R(C, X_0) & \longrightarrow & C \otimes_R \text{Hom}_R(C, M) & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \downarrow & & \\ X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and the vertical arrows are the natural evaluation maps. A diagram chase shows that the rightmost vertical arrow is an isomorphism, and so  $M \in \mathcal{B}_C(R)$ .  $\square$

**Lemma 5.7.** *If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for  $\mathcal{H}_C(\mathcal{F}_C(R))$ .*

*Proof.* The containment in the following sequence is from Facts 2.7 and 2.9

$$\mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$$

and the equality is from Proposition 5.4. Lemma 4.1 implies  $\mathcal{GF}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ . Thus, the conditions  $\mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \subseteq \mathcal{GF}_C(R)$  imply that we have  $\mathcal{H}_C(\mathcal{F}_C(R)) \perp \mathcal{F}_C^{\text{cot}}(R)$ .

Let  $M \in \mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{GF}_C(R)$ . Since  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for  $\mathcal{GF}_C(R)$  by Proposition 4.7, there is an exact sequence

$$0 \rightarrow M \rightarrow X \rightarrow M' \rightarrow 0$$

with  $X \in \mathcal{F}_C^{\text{cot}}(R)$  and  $M' \in \mathcal{GF}_C(R)$ . Since  $M$  and  $X$  are in  $\mathcal{B}_C(R)$ , Fact 2.7 implies that  $M' \in \mathcal{B}_C(R)$ . That is  $M' \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$ . This establishes the desired conclusion.  $\square$

**Lemma 5.8.** *If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{H}_C^2(\mathcal{F}_C(R)) \subseteq \text{cores } \widetilde{\mathcal{F}_C^{\text{cot}}(R)}$ .*

*Proof.* Lemma 5.7 says that  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator for  $\mathcal{H}_C(\mathcal{F}_C(R))$ . By Lemma 5.2(b) we know that  $\mathcal{H}_C^2(\mathcal{F}_C(R)) \perp \mathcal{F}_C^{\text{cot}}(R)$ . Let  $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$  and let  $X$  be a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{H}_C(\mathcal{F}_C)$ -resolution of  $M$ . By definition, the complex

$$0 \rightarrow M \rightarrow X_{-1} \rightarrow X_{-2} \rightarrow \cdots$$

is an augmented  $\mathcal{H}_C(\mathcal{F}_C)$ -coresolution that is  $\mathcal{F}_C$ -proper and therefore  $\mathcal{F}_C^{\text{cot}}$ -proper. Hence, Lemma 1.10 implies  $M \in \text{cores } \widetilde{\mathcal{F}_C^{\text{cot}}(R)}$ .  $\square$

**Theorem 5.9.** *For each semidualizing  $R$ -module  $C$  and each integer  $n \geq 1$ , there is an equality  $\mathcal{H}_C^n(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ .*

*Proof.* We first verify the equality  $\mathcal{H}_C^2(\mathcal{F}_C(R)) = \mathcal{H}_C(\mathcal{F}_C(R))$ . Remark 2.13 implies  $\mathcal{H}_C^2(\mathcal{F}_C(R)) \supseteq \mathcal{H}_C(\mathcal{F}_C(R))$ . For the reverse containment, let  $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$ . Lemma 3.3 implies  $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ , and so  $M \perp \mathcal{F}_C^{\text{cot}}(R)$  by Lemma 5.2(b). From Lemma 5.6 we have  $M \in \mathcal{B}_C(R)$ , and so Fact 2.7 provides an augmented  $\mathcal{P}_C$ -proper  $\mathcal{P}_C$ -resolution

$$(\ddagger) \quad \cdots \xrightarrow{\partial_2^Z} Z_1 \xrightarrow{\partial_1^Z} Z_0 \rightarrow M \rightarrow 0.$$

Each  $Z_i \in \mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ , so we have  $Z_i \perp \mathcal{F}_C^{\text{cot}}(R)$  by Lemma 3.3. We conclude from Lemma 1.9(a) that  $(\ddagger)$  is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact.

Lemma 5.8 yields a  $\mathcal{F}_C^{\text{cot}}$ -proper augmented  $\mathcal{F}_C^{\text{cot}}$ -coresolution

$$(\dagger) \quad 0 \rightarrow M \rightarrow Y_{-1} \rightarrow Y_{-2} \rightarrow \cdots$$

Since each  $Y_i \in \mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{B}_C(R)$  by Fact 2.7, we have  $C \perp Y_i$  for each  $i < 0$ , and similarly  $C \perp M$ . Thus, Lemma 1.9(b) implies that  $(\dagger)$  is  $\text{Hom}_R(C, -)$ -exact. It follows that the complex obtained by splicing the sequences  $(\ddagger)$  and  $(\dagger)$  is a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C$ -resolution of  $M$ . Thus, we have  $M \in \mathcal{H}_C(\mathcal{F}_C(R))$ .

To complete the proof, use the previous two paragraphs and argue by induction on  $n$  to verify the first equality in the next sequence

$$\mathcal{H}_C^n(\mathcal{F}_C(R)) = \mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R).$$

The second equality is from Proposition 5.4.  $\square$

Our next result contains Theorem II(a) from the introduction.

**Corollary 5.10.** *If  $C$  is a semidualizing  $R$ -module, then  $\mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$  for each  $n \geq 1$ .*

*Proof.* In the next sequence, the containments are from Fact 2.11 and Remark 2.13

$$\begin{aligned} \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) &\subseteq \mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{G}^n(\mathcal{H}_C(\mathcal{F}_C(R))) \\ &\subseteq \mathcal{H}_C^n(\mathcal{H}_C(\mathcal{F}_C(R))) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \end{aligned}$$

and the equalities are by Proposition 5.4 and Theorem 5.9.  $\square$

**Remark 5.11.** In light of Corollary 5.10, it is natural to ask whether we have  $\mathcal{G}(\mathcal{F}_C(R)) = \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$  for each semidualizing  $R$ -module  $C$ . While Remark 2.13 and Proposition 5.4 imply that  $\mathcal{G}(\mathcal{F}_C(R)) \subseteq \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$ , we do not know whether the reverse containment holds.

We now turn our attention to  $\mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R))$  and  $\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R))$ .

**Proposition 5.12.** *Let  $C$  be a semidualizing  $R$ -module and let  $n \geq 1$ .*

- (a) *We have  $\mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp \subseteq \mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R)) \subseteq \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$ .*
- (b) *If  $\dim(R) < \infty$ , then  $\mathcal{F}_C(R) \perp \mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R))$ .*
- (c) *If  $\dim(R) < \infty$ , then  $\mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$ .*

*Proof.* (a) For the first containment, let  $M \in \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$ . Since  $M \in \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$ , Lemma 3.5(c) yields an augmented  $\mathcal{F}_C^{\text{cot}}$ -resolution

$$\cdots \rightarrow Z_1 \rightarrow Z_0 \rightarrow M \rightarrow 0$$

that is  $\text{Hom}_R(C, -)$ -exact; the argument of Proposition 5.4 shows that this resolution is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact. Because  $M$  is in  $\mathcal{G}\mathcal{F}_C(R)$ , Proposition 4.7 provides an augmented  $\mathcal{F}_C^{\text{cot}}$ -coresolution

$$0 \rightarrow M \rightarrow Y_{-1} \rightarrow Y_{-2} \rightarrow \cdots$$

that is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact. Since  $M \in \mathcal{B}_C(R)$ , the proof of Proposition 5.4 shows that this coresolution is also  $\text{Hom}_R(C, -)$ -exact. Splicing these resolutions yields a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $M$ , and so  $M \in \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R)) \subseteq \mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R))$ .

The second containment follows from the next sequence

$$\mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R)) \subseteq \mathcal{H}_C^n(\mathcal{F}_C(R)) = \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$$

wherein the containment is by definition, and the equality is by Theorem 5.9.

(b) Assume  $d = \dim(R) < \infty$ . A result of Gruson and Raynaud [20, Seconde Partie, Thm. (3.2.6)] and Jensen [19, Prop. 6] implies  $\text{pd}_R(F) \leq d < \infty$  for each flat  $R$ -module  $F$ .

We prove the result for all  $n \geq 0$  by induction on  $n$ . The base case  $n = 0$  follows from Lemma 3.3. Assume  $n \geq 1$  and that  $\mathcal{F}_C(R) \perp \mathcal{H}_C^{n-1}(\mathcal{F}_C^{\text{cot}}(R))$ . Let  $M \in \mathcal{H}_C^n(\mathcal{F}_C^{\text{cot}}(R))$ , and let  $X$  be a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{H}_C^{n-1}(\mathcal{F}_C^{\text{cot}})$ -resolution of  $M$ . For each  $i$  set  $M_i = \text{Im}(\partial_i^X)$ . This yields an isomorphism  $M \cong M_0$  and, for each  $i$ , an exact sequence

$$0 \rightarrow M_{i+1} \rightarrow X_i \rightarrow M_i \rightarrow 0.$$

Note that  $M_i, X_i \in \mathcal{B}_C(R)$  by part (a). Let  $F \otimes_R C \in \mathcal{F}_C(R)$  and let  $t \geq 1$ . Since  $\mathcal{F}_C(R) \perp X_i$  for each  $i$ , a standard dimension-shifting argument yields the first isomorphism in the next sequence

$$\text{Ext}_R^t(F \otimes_R C, M) \cong \text{Ext}_R^{t+d}(F \otimes_R C, M_d) \cong \text{Ext}_R^{t+d}(F, \text{Hom}_R(C, M_d)) = 0.$$

The second isomorphism is a form of Hom-tensor adjointness using the fact that  $F$  is flat with the Bass class condition  $\text{Ext}_R^{\geq 1}(C, M_d) = 0$ . The vanishing follows from the inequality  $\text{pd}_R(F) \leq d$ .

(c) This follows from parts (a) and (b).  $\square$

**Lemma 5.13.** *Let  $C$  be a semidualizing  $R$ -module and assume  $\dim(R) < \infty$ . If  $M \in \mathcal{F}_C(R)$ , then  $\mathcal{F}_C^{\text{cot}}\text{-id}_R(M) \leq \dim(R) < \infty$ .*

*Proof.* Let  $F$  be a flat  $R$ -module such that  $M \cong F \otimes_R C$ . Since  $d = \dim(R)$  is finite, the flat module  $F$  has an  $\mathcal{F}_C^{\text{cot}}$ -coresolution  $X$  such that  $X_i = 0$  for all  $i < -d$ ; see [9, (8.5.12)]. Since  $M \in \mathcal{A}_C(R)$  and each  $X_i \in \mathcal{A}_C(R)$ , it follows readily that the complex  $X \otimes_R F$  is an  $\mathcal{F}_C^{\text{cot}}$ -coresolution of  $M$  of length at most  $d$ , as desired.  $\square$

Our final result contains Theorem II(b) from the introduction.

**Theorem 5.14.** *Let  $C$  be a semidualizing  $R$ -module and assume  $\dim(R) < \infty$ . Then  $\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$  for each  $n \geq 1$ , and  $\mathcal{F}_C^{\text{cot}}(R)$  is an injective cogenerator and a projective generator for  $\mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$ .*

*Proof.* We first show  $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) \supseteq \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$ . Let  $M \in \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$  and let  $X$  be a  $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$ -complete  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $M$ . To show that  $M$  is in  $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R))$ , it suffices to show that  $X$  is  $\text{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$ -exact, since it is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact by definition. For each  $i$ , set  $M_i = \text{Im}(\partial_i^X) \in \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$ . Lemma 3.3 and Proposition 5.12(b) imply  $\mathcal{F}_C(R) \perp X_i$  and  $\mathcal{F}_C(R) \perp M_i$  for all  $i$ . Hence, Lemma 1.9(b) implies that  $X$  is  $\text{Hom}_R(\mathcal{F}_C, -)$ -exact, and so  $X$  is  $\text{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$ -exact.

We next show  $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) \subseteq \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$ . Let  $N \in \mathcal{G}(\mathcal{F}_C^{\text{cot}}(R))$  and let  $Y$  be a complete  $\mathcal{F}_C^{\text{cot}}$ -resolution of  $N$ . We will show that  $Y$  is  $\text{Hom}_R(\mathcal{F}_C, -)$ -exact; the containment  $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$  will then imply that  $Y$  is  $\text{Hom}_R(\mathcal{P}_C, -)$ -exact. Since  $Y$  is  $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$ -exact by definition, we will then conclude that  $N$  is in  $\mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$ . We have  $\mathcal{F}_C(R) \perp Y_i$  for each  $i$  by Lemma 3.3, and so  $\mathcal{F}_C^{\text{cot}}(R) \perp Y_i$ . Since  $Y$  is  $\text{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$ -exact, Lemma 1.9(b) implies  $\mathcal{F}_C^{\text{cot}}(R) \perp M$ . From Lemma 1.8 we conclude that  $\text{cores } \widehat{\mathcal{F}_C^{\text{cot}}(R)} \perp M$ . Since  $\dim(R) < \infty$ , Lemma 5.13 implies that  $\mathcal{F}_C(R) \subseteq \text{cores } \widehat{\mathcal{F}_C^{\text{cot}}(R)}$  and so  $\mathcal{F}_C(R) \perp M$ . With the condition  $\mathcal{F}_C(R) \perp Y_i$  from above, this implies that  $Y$  is  $\text{Hom}_R(\mathcal{F}_C, -)$ -exact by Lemma 1.9(b).

The above paragraphs yield the second equality in the next sequence

$$\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp.$$

The first equality is from [23, (4.10)] since Lemma 3.3 implies  $\mathcal{F}_C^{\text{cot}}(R) \perp \mathcal{F}_C^{\text{cot}}(R)$ , and the third equality is from Proposition 5.12(c). The final conclusion follows from [23, (4.7)].  $\square$

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